A lower bound on the weight hierarchy of product codes

Hans Georg Schaathun
Institutt for Informatikk
Universitas Bergensis
Høyteknologisenteret
N-5020 Bergen, Norway
e-mail: georg@ii.uib.no

Wolfgang Willems
Fakultät für Mathematik
Otto-von-Guericke-Universität
D-39016 Madgeburg, Germany
e-mail:
wolfgang.willems@mathematik.uni-magdeburg.de

Abstract — **Let $C = C_1 \otimes \ldots \otimes C_t$ denote the product code of linear codes C_i. In this note we give a lower bound for the weight hierarchy of C in terms of the hierarchies of the components C_i. By [5] the bound is attained if all components are chained.**

Let $C = C_1 \otimes \ldots \otimes C_t$ denote the product code of linear $[n_i, k_i]$ codes C_i. Wei and Yang conjectured in 1993 a formula to compute the weight hierarchy of C in terms of the weight hierarchies of the components C_i in case that C consists of two chained componends. Over the last years much attention has been spent either to verify the formula in particular cases or to compute the formula for special components C_1 and C_2, see [8], [1], [4], [7], [5], [6].

The *weight hierarchy* of an $[n, k]$ code C is the sequence $(\mathrm{d}_1(C), d_2(C), \ldots, \mathrm{d}_k(C))$ where

$$\mathrm{d}_r(C) := \min\{\,|\operatorname{supp}(D)|\mid D \le C,\ \dim D = r\}$$

and

$$\operatorname{supp}(D) := \{\, i \mid \exists\, (d_1, \ldots, d_n) \in D \text{ with } d_i \neq 0\}.$$

Some authors call the weight hierarchy also *length/dimension profile*, see [3].

The code C is called *chained* if there exists a sequence of subcodes

$$D_1 < D_2 < \ldots < D_k = C$$

such that D_i has dimension i and support seize $\mathrm{d}_i(C)$.

Recently, the Wei-Yang conjecture has been proved independently by the first author [7], and Martínez-Pérez together with the second author [5]. Even more, in [7] a lower bound for the weight hierarchy of C is given for $t = 2$ and not necessarily chained components C_1 and C_2. Furthermore, the bound is attained if the codes are chained. Note here that this result does not extend inductively to the product of more than two chained components since the product of two chained components is usually not chained. Paper [5] gives an affirmative answer to the Wei-Yang conjecture even for arbitrary t. In this note we drop the assumption that the components have to be chained (a strong condition and for almost all codes not satisfied, see [2]) and generalize the lower bound for the weight hierarchy obtained in [7] to a product code with more than two components.

To be more precise, we define

$$\mathcal{M}_t := \{\mathbf{i} = (i_1, i_2, \ldots, i_{t-1}) \mid 1 \le i_j \le k_j, 1 \le j < t\}.$$

Let π be a map $\mathcal{M}_t \to \{0, 1, \ldots, k_t\}$ given by $\mathbf{i} \mapsto t_{\mathbf{i}}$. We call π a $(k_1, k_2, \ldots, k_t)$-partition of r if

1. $\sum_{\mathbf{i}\in\mathcal{M}_t} t_{\mathbf{i}} = r$ and
2. π is a non-increasing function in each coordinate, i.e. $t_{i_1,\ldots,i_j,\ldots,i_{t-1}} \le t_{i_1,\ldots,i_j-1,\ldots,i_{t-1}}$ for $j = 1, \ldots, t-1$ and $1 < i_j$.

Furthermore let

$$d_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) := \min\left\{\nabla(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)\right\},$$

where

$$\nabla(\pi) = \sum_{\mathbf{i}\in\mathcal{M}_t} \prod_{j=1}^{t-1} (d_{i_j}(C_j) - d_{i_j-1}(C_j)) d_{\pi(\mathbf{i})}(C_t).$$

The definition of d_r^* specializes to the Wei-Yang formula for $t = 2$.

With the above notations we have

Theorem. *If $C_1, C_2, \ldots, C_t$ are arbitrary linear codes, then*

$$d_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \ge d_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t).$$

By [5], the lower bound is attained if all components C_i are chained. This gives an affirmative answer to the Wei-Yang conjecture for a product code with more than two components.

The method we used to prove the theorem is based on projective multisets and the Segre embedding of projective spaces.

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